Applications of the Dieudonné Determinant*

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The determinant function has been studied for more than 175 years. Formerly proposed as a tool for solving simultaneous linear equations, the determinant is now recognized as useless for this practical chore. On the other hand, the determinant has many useful properties. It appears in exterior algebra; also as a mapping function from $\mathfrak{F}_n$ to $\mathfrak{F}$ (where $\mathfrak{F}$ is any field). Interesting inequalities involving the determinant function exist. The same function can be used to locate proper values of matrices, i.e., to define regions of exclusion.

Since its definition was published, very little in the way of applications of the Dieudonné determinant has appeared. In this article we derive certain properties of the (ordinary) determinant function, and extend these properties to the Dieudonné determinant, a function defined on matrices over a skew field (division ring). The properties are extensive enough to permit new applications. In particular we show how to define the permanent function of a matrix over a division ring. On the other hand, the range of applications given here could undoubtedly be extended still further.

Some of the applications we adduce lead to new results. Others are recoveries of results that may perhaps have already been discovered in an even more general context. Nevertheless the interconnections expounded will be instructive and, we hope, of interest.

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* To Alexandre Ostrowski on his 75th birthday.

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1. PRELIMINARIES

In our view, all properties of the determinant function stem from Lemma 1.2, which states that a matrix $A \in \mathfrak{F}_n$ can be factored into a product of elementary matrices.

1.1. Definition. An elementary matrix is a matrix

$$T_{ij} = I + e_{ij} \qquad \text{or} \qquad S_{a,i} = I + (a-1)e_{ii}.$$

Note that every matrix T_{ij} is invertible, and that $S_{a,i}$ is invertible if a is invertible.

The main stream of our discussion concerns the ring of n-dimensional matrices over a skew field $\mathfrak{F}$.

1.2. Lemma. *Every matrix $A \in \mathfrak{F}_n$ is expressible as a product of elementary matrices.*

The product is of course not uniquely determined; but we shall use Lemma 1.2 to define the determinant of a matrix.

1.3. Definition. A determinant function over the matrices of $\mathfrak{F}_n$ is a function "det" from $\mathfrak{F}_n$ into $\mathfrak{F}$ (see below) with one of the following properties:

$$\left.\begin{array}{l}1.4.\\ \\ 1.5.\end{array}\right\} \det(AB) = (\det A)(\det B) = (\det B)(\det A) \begin{cases} \text{for every invertible } A, B \in \mathfrak{F}_n \\ \text{for every } A, B \in \mathfrak{F}_n \end{cases}$$

We recognize that the constant functions 1, 0 satisfy these definitions; these are trivial. The function that is 1 on the invertible matrices and 0 on the singular matrices satisfies property 1.5. It has been discovered many times, but we consider it also trivial. For the existence of a nontrivial determinant function, it is necessary that $\mathfrak{F}$ have more than two elements.

We first show that properties 1.4 and 1.5 are essentially equivalent.

1.6. Theorem. *Let* det *be a nontrivial determinant function satisfying property* 1.5. *Then* det *is identically* 0 *for any singular matrix.*

Proof. If O is the zero matrix and A is a matrix such that $\det A \neq 0, 1$, then $\det O = (\det A)(\det O)$ because $O = A \cdot O$. Therefore $\det O = 0$. Next, note that $\det I = 1$ since $IA = A$. Further, the determinant of a permutation matrix P is not 0, since $P^m = I$ for some m. It is now easy to see that $\det \operatorname{diag}[1, 1, \ldots, 1, 0, 0, \ldots, 0]$ must be 0 if the matrix is singular; this follows from the fact that O is expressible as the product of this matrix by some of its permutes. Now, if B is singular, B can be written as a product CDG, where D is the above matrix. Hence, $\det B = 0$.

We are justified in considering only property 1.4 henceforth, that is, only the mappings from the general linear group $GL[n, \mathfrak{F}]$ into $\mathfrak{F}$.

2. THE DETERMINANT OF A 1×1 MATRIX

We discuss the possible existence of a nontrivial determinant function for 1×1 matrices, the simplest case. First we note that $\det a$ is never 0; hence, $\det 1 = 1$ because $\det(1 \cdot a) = (\det 1)(\det a)$. Further, since $\det a \det(a^{-1}) = \det 1$, it follows that $(\det a)^{-1} = \det(a^{-1})$ and finally from $(\det a)(\det b) = (\det b)(\det a)$ it follows that the determinant of any commutator $aba^{-1}b^{-1}$ is 1. Let K^* be the multiplicative group of $\mathfrak{F}$; then det is a mapping from K^* into the quotient group $K^*/K^{*\prime}$ of K^* with respect to its commutator subgroup. The kernel of this mapping is clearly the commutator subgroup $K^{*\prime}$, and the mapping is completely defined. As we shall see later, this conclusion, derived in the 1×1 case, can be generalized to $GL[n, \mathfrak{F}]$. Note that $K^*/K^{*\prime} \simeq C^*$, the center of K^*. The determinant of an $n \times n$ matrix will be defined as an element of $K^*/K^{*\prime}$, i.e., as a coset of $K^{*\prime}$ in K^*. It will not always be possible to replace the entire coset by a single representative. When K^* is commutative, there is hardly any distinction, since every coset has only one member.

Let us digress to consider how a determinant function must be defined over 1×1 matrices when the coefficient domain is a ring but not necessarily a skew field. First, if the ring has no zero divisors, e.g., if the ring is a ring of polynomials over a division ring, it can be imbedded in a quotient field [24]. A self-consistent determinant function is thus immediately defined. In the general case when zero divisors are present, it may be useful to define the determinant function only after reducing the ring modulo its radical (see [17, p. 221]). However, this device is not always effective.

Remark. The matrix equation

$$\begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix} \begin{bmatrix} \beta & 0 \\ 0 & \beta^{-1} \end{bmatrix} = \begin{bmatrix} \alpha\beta & 0 & \\ 0 & \alpha^{-1} & \beta^{-1} \end{bmatrix}$$

shows the desirability of Definitions 1.4 and 1.5. The product $\alpha\beta\alpha^{-1}\beta^{-1}$ should probably be indistinguishable from 1; i.e., $\det B \det A = \det A \det B$.

3. THE DETERMINANT FUNCTION FOR $n \times n$ MATRICES

3.1. LEMMA. *If $n > 2$, the matrix T_{ij} is a commutator (if $i \neq j$).*

Proof. $T_{ik}T_{kj}T_{ik}^{-1}T_{kj}^{-1} = T_{ij}$ (if $i \neq k \neq j$).

3.2. LEMMA. *If t, $t - 1$ are both invertible, the matrix T_{12} is a commutator.*

Proof.

$$\begin{bmatrix} t & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & (t-1)^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} t^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -(t-1)^{-1} \\ 0 & 1 \end{bmatrix} = T_{12}.$$

In particular T_{12} is a commutator whenever 2 is invertible ($t = -1$).

3.3. LEMMA. *The matrix* $\begin{bmatrix} 1 & 2a \\ 0 & 1 \end{bmatrix}$ *is a commutator.*

Proof.

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2a \\ 0 & 1 \end{bmatrix}.$$

LEMMA. *The matrix* $\begin{bmatrix} a & \\ & a^{-1} \end{bmatrix}$ *is a product of commutators ($a \neq 0$).*

3.4. *Proof.* If a, $a - 1$ are both invertible, then each factor in the product is a commutator:

$$\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} = \begin{bmatrix} 1 & a(a-1) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ a^{-1}(1-a) & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ a-1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

3.5. *Remark.* If $\mathfrak{F}$ has two elements, $n = 2$, the matrices T_{12}, T_{21} are not products of commutators. For the six nonsingular matrices of $\mathfrak{F}_2$ are I, $A = T_{12}$, $B = T_{21}$, $BA = ABAB$, $AB = BABA$, $ABA = BAB$, and the commutator subgroup is $\{I, BA, AB\}$. But since $\mathfrak{F}$ has only two elements, no nontrivial determinant function can exist.

3.6. DEFINITION. Any matrix $\operatorname{diag}[a, 1, 1, \ldots, 1]$, $a \neq 0$, is called a *determining* matrix.

3.7. THEOREM. *Every invertible matrix can be written as the product of a determining matrix (first factor) times a product of commutators (second factor).*

Proof. The proof bases itself on Lemmas 1.2, 3.1, 3.2, 3.3. Write the given matrix as a product of elementary matrices, and insert extra factors, e.g.,

$$T\begin{bmatrix}1 & 0\\ 0 & a\end{bmatrix} = T\begin{bmatrix}a^{-1} & 0\\ 0 & a\end{bmatrix}\begin{bmatrix}a & 0\\ 0 & 1\end{bmatrix}; \qquad U\begin{bmatrix}a & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}a & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}a & 0\\ 0 & 1\end{bmatrix}^{-1} U \begin{bmatrix}a & 0\\ 0 & 1\end{bmatrix},$$

where T, U are arbitrary products of commutators. Since the commutator subgroup $CGL[n, \mathfrak{F}]$ is invariant, the theorem follows.

For a nontrivial map we easily derive $\det I = 1$, $(\det A)^{-1} = \det A^{-1}$, and the determinant of a commutator must be 1. Hence from property 1.5, the determinant of any matrix must be the determinant of its determining matrix. We *define* the map of $\operatorname{diag}[a, 1, \ldots, 1]$ as the mapping from K^* onto $K^*/K^{*\prime}$ (or by abuse of language, onto C^*) which was given earlier (with kernel $K^{*\prime}$). This function is the determinant function defined by Dieudonné. In case $\mathfrak{F}$ is commutative, it coincides with the usual determinant function.

In my opinion, the above discussion is more direct than most. It was given in essentially the same form by Taussky and Wielandt in 1963, in my lectures in 1955, and perhaps earlier by others.

Some properties of this definition have to be established.

3.8. THEOREM. *The "determinant of the determining matrix" is a determinant function satisfying property* 1.5.

Proof. Suppose

$$A = \begin{bmatrix}a & 0\\ 0 & I\end{bmatrix} T, \qquad B = \begin{bmatrix}b & 0\\ 0 & I\end{bmatrix} U.$$

Then

$$AB = \begin{bmatrix} ab & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} b^{-1} & 0 \\ 0 & I \end{bmatrix} T \begin{bmatrix} b & 0 \\ 0 & I \end{bmatrix} U$$

has determinant $\det(ab) = (\det A)(\det B)$. (We have used $\det_1 ab$ to mean the mapping from K^* onto C^*, and $\det_n A$ to mean the mapping from $GL[n, \mathfrak{F}]$ onto C^*.)

3.9. THEOREM. *The determinant of a matrix is the same as the determinant of its transpose.*

Proof. The factorization of a matrix into a product of elementary matrices establishes this. Note that the transpose of a product is the product of the transposes in reverse order.

3.10. THEOREM. *If one row of a matrix is multiplied by the constant a, the determinant is multiplied by this same constant.*

Proof. The matrix transformation in question amounts to multiplication of the matrix by an elementary matrix $I + (a - 1)e_{ii}$, the determinant of which is a.

3.11. THEOREM. *If two rows of a matrix are interchanged and afterward one of these rows is multiplied by -1, the determinant is unaltered.*

Proof. The matrix transformation is brought about by premultiplying by the matrix

$$\begin{bmatrix} 0, & 1 \\ -1, & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Each of the three displayed factors is a commutator.

3.12. COROLLARY. *Interchange of two rows of a matrix changes the determinant to its negative.*

Here $-\det_1 a$ means $\det_1(-a)$. Over quaternions, $\det(-a)$ and $\det a$ are the same.

Proof. Theorems 3.10 and 3.11.

3.13. THEOREM. *If one row of a matrix is augmented by a multiple of another row, the determinant is unchanged.*

Proof. The transformation amounts to premultiplication by

$$\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}.$$

3.14. LEMMA. *Suppose* $A = \begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix}$, *i.e., A is obtained from B by bordering with an identity matrix.* *Then* $\det A = \det B$.

3.15. DEFINITION. The matrix $A = [a_{ij}]$ is a direct sum: $A = B \oplus C$, if the indices $\{1, \ldots, n\}$ can be partitioned into disjoint sets R, S such that $a_{ij} = 0$ if $i \in R$ and $j \in S$ and also $a_{ij} = 0$ if $i \in R$ and $j \in S$. The matrix B is $B = [a_{ij}]_{i,j \in R}$; $C = [a_{ij}]_{i,j \in S}$. $A \approx \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$.

3.16. THEOREM. $\det B \oplus C = (\det B)(\det C)(\pm 1)$, *where the factor* ± 1 *indicates the sign of the permutation* $\{1, \ldots, n\} \to \{R, S\}$.

Proof. Note that $B \oplus C = [B \oplus I][I \oplus C]$, where the symbol $\oplus$ denotes direct sum. Express $[B \oplus I]$ and $[I \oplus C]$ as products of $n \times n$ elementary matrices ||.

3.17. DEFINITION. The matrix $A = [a_{ij}]$ is reducible if the indices $\{1, \ldots, n\}$ can be partitioned into disjoint sets R, S such that $a_{ij} = 0$ if $i \in R$ and $j \in S$.

3.18. THEOREM. *Let A be reducible; let B, C be defined as in Lemma* 3.14. *Then* $\det A = (\det B)(\det C)(\pm 1)$.

Proof. One first shows that $A = D[B \oplus C]$, where D is a product of commutators; Theorem 3.18 then follows from Definition 3.17 and Theorem 3.8.

4. COMBINATORIAL PROPERTIES OF THE DETERMINANT FUNCTION

In the preceding section the determinant function was defined; certain properties were shown to be immediate consequences of the definition. When properly rephrased, any known property of the determinant function over commutative domains can be carried over to the noncommutative case. In this section we study chiefly the generalizations of multilinearity.

It has to be remembered that the determinant of the 1×1 matrix $[a]$ is not a, but is the coset of K^* modulo the commutator group $K^{*\prime}$ to which a belongs. When only multiplications are being performed, each coset may be represented by an element of C^*, since multiplication of cosets amounts to multiplication of their representatives. This simplified representation does not work if addition is involved as well as multiplication.

If one overlooks this fact, one overlooks at the same time the possibility of expanding the determinant of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $(a \neq 0)$. Because of the relation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & -a^{-1}b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & d - ca^{-1}b \end{bmatrix}$$

noticed by many early writers, it must be so that

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a(d - ca^{-1}b) = ad - aca^{-1}b.$$

The relation between this number and the number $ad - bc$ is the following.

4.1. Lemma. *There are elements U, V, W from the commutator group $K^{*\prime}$ of K^* such that*

$$ad - Wcb = \det A,$$

$$ad - cbU = \det A,$$

$$ad - bcV = \det A.$$

Proof. $W = aca^{-1}c^{-1}$, $U = (cb)^{-1}Wcb$, $V = c^{-1}b^{-1}cbU$.

This is not an artificiality. We must think not $\det A = ad - bc$, but rather $\det A = a \cdot \det d - b \cdot \det c$; and $\det d$, $\det c$ are determined only up to a factor from the commutator group $K^{*\prime}$. (In the commutative case, $K^{*\prime}$ is trivial.) Before establishing the noncommutative form of the multilinearity property, we interpose a short digression.

4.2. LEMMA. *The following properties of the determinant function are equivalent:*

(a) *Whenever the first row of matrix A is the sum of two rows $r_1 + r_2$,*

$$A = \begin{bmatrix} r_1 + r_2 \\ \cdots \\ \cdots \end{bmatrix},$$

then $\det A = \det A_1 + \det A_2$, *where A_i is the same as A except in the first row, where r_i replaces $r_1 + r_2$.*

(b) *Whenever the first column of A is the vector $[1, -1, 0, \ldots, 0]^*$, the determinant of A can be found by expanding by minors of the first column.*

(c) *The determinant of every matrix can be found by expanding by minors.*

Proof. To show parts (a) and (b) are equivalent, consider the bordered matrix

$$\begin{bmatrix} 1 & r_1 \\ -1 & r_2 \\ 0 & \cdots \\ \vdots & \\ 0 & \cdots \end{bmatrix}$$

and expand it in two ways: first, by minors of the first column; second, by adding the first row to the second row and using Theorem **3.18**.

To show that parts (a) and (c) are equivalent, note that part (c) amounts to applying (a) inductively, by considering the pivot row as the sum of n rows, viz,

$$[\alpha, \beta, \gamma, \ldots] = [\alpha, 0, \ldots, 0] + [0, \beta, 0, \ldots] + [0, 0, \gamma, \ldots] + \cdots.$$

4.3. THEOREM. *The determinant of an $n \times n$ matrix can be found by expanding by minors of any one row or column.*

Proof. The inductive proof assumes that parts (a) and (c) of Lemma 4.2 are valid for matrices of order less than n, and also that Theorem **4.3** is valid for any matrix having $k - 1$ or fewer nonzero elements in the pivot row. The induction must then be carried to an $n \times n$ matrix with exactly k nonzero elements in one row. To make the exposition readable, we simply expound the proof for the 3×3 case:

$$A = [a_{ij}]_1^3, \qquad A \begin{bmatrix} 1 & 0 & -a_{11}^{-1}a_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12}, & 0 \\ a_{21} & a_{22}, & a_{23} - a_{21}a_{11}^{-1}a_{13} \\ a_{31} & a_{32}, & a_{33} - a_{31}a_{11}^{-1}a_{13} \end{bmatrix} = A_1.$$

Since $\det A = \det A_1$, the induction hypothesis implies that

4.4.

$$\det A = a_{11} \det \begin{bmatrix} a_{22} & a_{23} - a_{21}a_{11}^{-1}a_{13} \\ a_{32} & a_{33} - a_{31}a_{11}^{-1}a_{13} \end{bmatrix} - a_{12} \det \begin{bmatrix} a_{21} & a_{23} - a_{21}a_{11}^{-1}a_{13} \\ a_{31} & a_{33} - a_{31}a_{11}^{-1}a_{13} \end{bmatrix}.$$

The usual theorems are valid for matrices of lower order by the induction hypothesis. Thus

$$a_{11} \det \begin{bmatrix} a_{22} & a_{23} - a_{21}a_{11}^{-1}a_{13} \\ a_{32} & a_{33} - a_{31}a_{11}^{-1}a_{13} \end{bmatrix} = a_{11} \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

$$+ a_{11} \det(a_{11}^{-1} \quad a_{13}) \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

where we used

$$\begin{bmatrix} a_{22} & a_{21}a_{11}^{-1}a_{13} \\ a_{32} & a_{31}a_{11}a_{13} \end{bmatrix} = \begin{bmatrix} a_{22} & a_{21} \\ a_{32} & a_{31} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & a_{11}^{-1}a_{13} \end{bmatrix}.$$

The proof of Theorem 4.3 may now be completed by treating the last term of the equation 4.4 in similar fashion. The proof for matrices of arbitrary dimension is the same, with the necessary tedious generality of notation.

Remark. Theorem 4.3, stated in the form (a) of Lemma 4.2, was discovered by W. Givens [12]. His proof seems not to have been published.

Remark. It is now clear that assertions (a), (b), (c) under Lemma 4.2 are not only equivalent, but are in fact universally valid.

4.5. THEOREM (*Laplace expansion*). *The generalized expansion*

$$\det A = \sum \pm \det A \begin{pmatrix} 1 \cdots r \\ k_1 \cdots k_r \end{pmatrix} \det A \begin{pmatrix} r+1 \cdots n \\ \{1, \ldots, n\} \setminus \{k_1 \cdots k_r\} \end{pmatrix}$$

is valid.

Here, the notation $A\ (\ast\ast\ast)$ denotes the minor based on rows ... and columns $\ast\ast\ast$; the set $\{1, \ldots, n\} \setminus \{k_1 \cdots k_r\}$ is the set $\{1, \ldots, n\}$ with the element $k_1, \ldots, k_r$ deleted; and the summation is extended over all subsets $\{k_1, \ldots, k_r\}$ of r indices from the set $\{1, \ldots, n\}$. The proof is again by induction.

4.6. THEOREM (*Cramer's rule*). *The solution of the linear system* $Ax = b$ *satisfies the usual rules; instead of* $x = A^{-1}b$, *however, we must write* $(\det x_i) = \det(A^{-1}b)_i$. *Conditions for solvability, number of linearly independent solutions, etc. remain the usual ones, it being understood that "solution" means "solution coset."*

Remark. In a field with valuation, the relation $x = A^{-1}b$ can almost be achieved; in fact $\|x_i\| = \|(A^{-1}b)_i\|$. The value of a commutator is 1.

5. COMPOUND MATRICES

The compound of a matrix can be defined in the usual way. The elements of the compound are themselves cosets of $K^{*\prime}$. Note that a matrix and its first compound are not identical; for matrices over a commutative field, there is no need to distinguish between them.

5.1. DEFINITION. Let $A = [a_{ij}]$ be an $n \times m$ matrix. The rth compound $A(r)$ of A $(1 \leqslant r \leqslant \min(n, m))$ is the $\binom{n}{r} \times \binom{m}{r}$ matrix, the elements of which are the determinants of the various $r \times r$ minor matrices of A, written in lexicographic order by rows and columns.

5.2. THEOREM. *If* A, B *are any matrices (for which* AB *is defined), then* $A^{(r)}B^{(r)} = (AB)^{(r)}$.

The assertion is less precise in the noncommutative case than in the commutative case; the idea is that a purported relation such as $cd +$

$ef + \cdots = gk$ holds if each letter is a suitable representative of its coset. In the commutative case, each coset has only one representative.

Proof. By Lemma 1.2, it is sufficient to establish Theorem 5.2 in the special case that B is an elementary matrix. For if B is merely the product B_1B_2 of two elementary matrices, then

$$(AB)^{(r)} = (AB_1)^{(r)}B_2^{(r)} = A^{(r)}B_1^{(r)}B_2^{(r)} = A^{(r)}(B_1B_2)^{(r)} = A^{(r)}B^{(r)}.$$

The formal inductive proof assumes Theorem 5.2 to be valid whenever B is the product of $i - 1$ elementary matrices, and on the basis of this assumption, establishes the theorem when B is the product of i elementary matrices.

Theorem 5.2 is obvious when B is an elementary matrix T_{ij}; see property (a) under Lemma 4.2. If B is $S_{a,i}$, Theorem 5.2 is also obvious; see Theorem 3.10.

Remark. The above proof seems to be different from the proofs usually given, even in the commutative case.

5.3. LEMMA. *Set* $A = T_{ij}$, $i \neq j$. *Then* $\det A = 1$, $\det A^{(r)} = 1$.

Proof. From the definition of $A^{(r)}$ it is obvious that $A^{(r)}$ is upper triangular and has diagonal entries all equal to 1.

5.4. COROLLARY. *Set* $A = I + \alpha e_{ij}, i \neq j$. *Then* $\det A = 1, \det A^{(r)} = 1$.

The above proof applies.

5.5. LEMMA. *Let* $A = S_{a,i}$, *the diagonal matrix of order* n *with* a *in the* (i, i) *position. Then* $\det A^{(r)} = a$, *raised to power* $\binom{n-1}{r-1}$.

Proof. All nonprincipal minors of A have a row (or a column) of zeros. Every principal minor is either the identity matrix, or else has a as one diagonal entry. The number of the latter is $\binom{n-1}{r-1}$.

5.6. THEOREM. $\det A^{(r)} = (\det A)$, *raised to power* $\binom{n-1}{r-1}$.

Proof. Lemma 1.2, Theorem 5.2, Lemmas 5.3, 5.5.

Sylvester's determinant theorem. A theorem of Sylvester gives the values of certain principal minors of $A^{(r)}$; we write $A = [a_{ij}]_1^n$; $\Delta_s = \det [a_{ij}]_1^s$, where s is fixed, $1 \leqslant s \leqslant r$.

5.7. THEOREM. *Let B be the matrix, the elements of which consist of the determinants of all those $r \times r$ minors of A that involve the first s rows, the first s columns, (and $r - s$ other rows, $r - s$ other columns); elements of B are arranged according to the lexicographic order of these minors. Then* $\det B = (\det A)^\alpha (\det A_1)^\beta$, *where*

$$\alpha = \binom{n-s-1}{r-s-1}, \qquad A_1 = A\begin{pmatrix}1 \cdots s\\ 1 \cdots s\end{pmatrix}, \qquad \beta = \binom{n-s-1}{r-s}.$$

The proof is essentially that in [22].

6. HYBRID THEOREMS

One of the early hybrid theorems is due to Ingraham. The theorem concerns an $nr \times nr$ matrix $A = [a_{ij}]_1^{nr}$ that is partitioned into blocks $[A_{\mu\nu}]_1^n$ of equal size: $A_{\mu\nu} = [a_{ij}]$, $(\mu - 1)r < i \leqslant \mu r$, $(\nu - 1)r < j \leqslant \nu r$. Ingraham proved the theorem under the double assumption that all submatrices $A_{\mu\nu}$ are commutative, and that the field of coefficients is also commutative. See [16]. Theorem 6.1 includes Ingraham's theorem as a special case.

6.1. THEOREM. *Let $A = [a_{ij}]_1^{nr}$ be partitioned into n^2 equally sized $(r \times r)$ blocks $[A_{\mu\nu}]_1^n$. Then* $\det_{nr} A = \det_r(\det_n A)$.

The theorem says, for example, that

$$\det_{2r}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \det_r(A_{11}A_{22} - A_{12}A_{21});$$

but if $A_{\mu\nu}$ are not mutually commutative, this must be modified to read $\det_{2r} A = \det_r(A_{11}A_{22} - A_{12}A_{21}W)$, valid if W is a suitably chosen member of the commutator subgroup of the multiplicative group generated by $A_{\mu\nu}$.

Remark. The preceding paragraph is expository only. The determinant of $\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ is simply a mapping from 2×2 matrices with elements from $\mathfrak{F}_r$ into $K_r^*/K_r^{*\prime}$ itself. (The formula $A_{11}A_{22} - A_{12}A_{21}$ is not fundamental to the existence of this mapping.) We must know, however, that such a mapping can be defined. This is surely the case if the matrices $A_{\mu\nu}$ are all invertible. In the special case considered by Ingraham, the restriction to invertible submatrices is put aside as follows. Each $A_{\mu\nu}$ is replaced by a matrix $B_{\mu\nu} = A_{\mu\nu} - \lambda I$ of the same dimension. Except for a finite number of values of λ, all $B_{\mu\nu}$ are invertible, and the theorem is established with $B_{\mu\nu}$ in place of $A_{\mu\nu}$. The concluding step of the argument (descent from $B_{\mu\nu}$ to $A_{\mu\nu}$ by setting $\lambda = 0$) depends on properties of polynomials over the various domains that are involved. The validity of this step must be investigated for each individual domain $\mathfrak{F}_{nr}$, $\mathfrak{F}_r$, $\mathfrak{F}_n$. If all domains are commutative, there is no problem. Otherwise, the invertibility of all $A_{\mu\nu}$ seems to be an essential hypothesis.

Proof. The proof is essentially the same as Ingraham's, so the latter came within an ace of discovering the noncommutative determinant function. The formalisms in the proof of Theorems **4.3** and **3.18** explain how an inductive proof can be worded. The details are omitted.

6.2. COROLLARY. *Let A be a matrix of complex numbers: $A = [a_{ij}]$. Set* $\operatorname{Re} a_{ij} = g_{ij}$, $\operatorname{Im} a_{ij} = h_{ij}$; $a_{ij} = g_{ij} + h_{ij}\sqrt{-1}$. *Replace each entry a_{ij} by the 2×2 matrix $\begin{bmatrix} g_{ij} & h_{ij} \\ -h_{ij} & g_{ij} \end{bmatrix}$, thus expanding A to a $2n \times 2n$ real matrix G. Then* $|\det A|^2 = \det G$.

6.3. COROLLARY. *Let A be a matrix of quaternions; expand A in the same way into a $4n \times 4n$ real matrix G. Then* $|\det A|^4 = \det G$.

These corollaries indicate (in principle) a method of finding the real and imaginary parts of the roots of a complex or quaternion matrix by adhering to real arithmetic.

7. PROPER VALUES

The study of invariant subspaces and proper values can be carried quite far even over a noncommutative division ring.

7.1. DEFINITION. The scalar $\lambda \in \mathfrak{F}$ is called a (right) proper value of the matrix $A \in \mathfrak{F}_n$ if for some nonzero vector x the relation $Ax = x\lambda$

holds. It will appear that there is no distinction between right and left proper values.

An $n \times n$ matrix may fail to have proper values, or it may have an infinite number of them. The product $x\lambda$ represents the matrix operation of multiplying a column by a 1×1 matrix.

7.2. THEOREM. *If λ is a proper value corresponding to the vector x, then $\rho^{-1}\lambda\rho$ is a proper value corresponding to $x\rho$.*

7.3. THEOREM. *If x is a proper vector of A then $y = Px$ is a proper vector of PAP^{-1}.*

Proof. $\{Ax = x\lambda\} \Rightarrow Ax\rho = x\rho(\rho^{-1}\lambda\rho)$; $PAP^{-1}y = y\lambda$.

7.4. DEFINITION. The division ring $\mathfrak{F}$ has property $pv(n)$ [proper values up to n] if every matrix in $\mathfrak{F}_n, \mathfrak{F}_{n-1}, \ldots, \mathfrak{F}$ has a proper value. Clearly $\mathfrak{F}$ has property $pv(1)$ always.

7.5. THEOREM. *If $\mathfrak{F}$ has property $pv(n)$, then every matrix $A \in \mathfrak{F}_n$ is similar to a triangular matrix $B = [b_{ij}]$, i.e., $b_{ij} = 0$ if $i > j$.*

Proof. The proof goes by induction on n; i.e., we assume the theorem to be true for a matrix $C \in \mathfrak{F}_{n-1}$. It is only necessary to notice that a vector can always be bordered to give an invertible matrix. By Theorem 7.3 we can assume $x_1 \neq 0$, and state that $X = \begin{bmatrix} x_1 & 0 \\ z & I \end{bmatrix}$ has the inverse $\begin{bmatrix} x_1^{-1}, & 0 \\ -zx_1^{-1}, & I \end{bmatrix}$. Thus if $Ax = x\lambda$, then $AX = X \begin{bmatrix} \lambda, & w \\ 0, & c \end{bmatrix}$, where $w = [x_1^{-1}a_{1j}]$; $c_{ij} = -z_i x_1^{-1} a_{1j} + a_{ij}$. Here $z_i = x_{i+1}$. The necessary inductive step is established.

Remark. Unitarity need not be defined in $\mathfrak{F}$, so we cannot assert that A can be unitarily transformed to diagonal form.

7.6. THEOREM. *A matrix $A \in \mathfrak{F}_n$ has no more than n (dissimilar) proper values.*

Proof. Using Theorem 7.3 and the method of Theorem 7.5, we may replace A by the triangular matrix $B = XBX^{-1}$. We show that the proper values of a triangular matrix are its diagonal elements and the numbers similar to them. If $Bx = x\lambda$ and $x_1 \neq 0$, then $\lambda = x_1^{-1}b_{11}x_1$. If $x_1 = \cdots = x_{k-1} = 0$, then $\lambda = x_k^{-1}b_{kk}x_k$.

7.7. THEOREM. *If x is a nonzero vector and $Bx = 0$ $[x^*B = 0]$ then B is not invertible.*

Proof. If B^{-1} existed, then $B^{-1}Bx = x$ $[x^*BB^{-1} = x^*]$ would be 0.

7.8. THEOREM. *If λ is a [right] proper value of A, then $A - \lambda I$ has zero determinant and conversely.* (It is assumed that $\mathfrak{F}$ is a division ring).

Proof. $\{Ax = x\lambda\} \rightleftharpoons \{Ax = (\lambda I)x\} \rightleftharpoons \{(A - \lambda I)x = 0\}$.

7.9. THEOREM. *Every right proper value of A is a left proper value.* (It is not necessary to distinguish between right and left proper values.)

7.10. COROLLARY. *The proper values of a matrix and those of its transpose are the same.*

7.11. THEOREM. *Let* $A = \begin{bmatrix} A_{11}, & A_{12} \\ 0, & A_{22} \end{bmatrix}$ *have block triangular form, i.e., suppose A_{11}, A_{22} are square. Every proper value of A_{11} $[A_{22}]$ is a proper value of A. Every proper value of A is a proper value either of A_{11} or of A_{22}.*

Proof. If $Ax = x\lambda$, then $A_{11}z + A_{12}w = z\lambda$, $A_{22}w = w\lambda$, where $x = [z, w]^*$. If $w \neq 0$, λ is a proper value of A_{22}; if $w = 0$, λ is a proper value of A_{11}. The converse is immediate.

7.12. THEOREM. *If $\mathfrak{F}$ has property $pv(n)$, then every matrix $A \in \mathfrak{F}_n$ is similar to a matrix* $\operatorname{diag}[B_{11}, B_{22}, \ldots, B_{rr}]$, *where each matrix $B_{\mu\mu}$ is triangular with constant diagonal entries.*

Proof. Let $A = \begin{bmatrix} A_{11}, & C \\ 0, & B \end{bmatrix}$, where the proper values of A_{11} are all similar and none of these is a proper value of B. If we can solve the equation $A_{11}Z - ZB = C$, the proof is completed on transforming A by

$\begin{bmatrix} I & Z \\ 0 & I \end{bmatrix}$. We may assume that A_{11}, B are triangular. In this case, the equations to be solved are

$$
\begin{aligned}
a_{11}z_{11} + a_{12}z_{12} + \cdots + a_{1\mu}z_{\mu 1} - z_{11}b_{11} &= c_{11}, \\
a_{22}z_{21} + \cdots + a_{2\mu}z_{\mu 1} - z_{21}b_{11} &= c_{21}, \\
&\vdots \\
a_{\mu\mu}z_{\mu 1} - z_{\mu 1}b_{11} &= c_{\mu 1},
\end{aligned}
$$

together with further equations that concern the later columns of C. By a theorem of [21], these μ equations can be solved for z_{i1}, solving the last one first. The theorem is proved.

7.13. COROLLARY. *Let A, B be square matrices each of dimension not exceeding n and suppose $\mathfrak{F}$ has property $pv(n)$. Then the matrix equation $AZ - ZB = C$ is solvable provided that the proper values of A, B are disjoint.*

Proof. The given equation can be written in the form $SAS^{-1}(SZT) + (SZT)T^{-1}BT = SCT$. Thus we may assume that A, B are in triangular form and proceed as in Theorem 7.12. The corollary has the following paraphrase: There exists Z such that $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ can be transformed into block diagonal form by $\begin{bmatrix} I & Z \\ 0 & I \end{bmatrix}$.

7.14. THEOREM. *Let $\mathfrak{F}$ be a division ring. Every matrix $A \in \mathfrak{F}_n$ can be transformed (rationally) into almost triangular form (i.e., $i > j + 1 \Rightarrow a_{ij} = 0$).*

This theorem is well known to numerical analysts, who use the term Hessenberg form.

Proof. As usual, we use induction on n. Either $a_{21} = a_{31} = \cdots = a_{n1} = 0$ or else we arrange by a preliminary permutation that $a_{21} \neq 0$. The inductive step is completed by means of elementary transformations, the transforming matrices being $I + a_{21}^{-1}a_{j1}e_{j2}$.

It may occur that $a_{i+1,i} = 0$ for certain indices. Such an event signals decomposition into block triangular form. We study one of the blocks.

Thus, we assume (renaming) that A is a matrix in almost triangular form; $\forall_i\{a_{i+1,i} \neq 0\}$; $\forall_{i,j}\{(i > j+1) \Rightarrow a_{ij} = 0\}$. See Theorem 7.11.

We first remark that from $Ax = x\lambda$ there follows $x_n \neq 0$. Indeed the relation $Ax = x\lambda$ reads:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = x_1\lambda,$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = x_2\lambda,$$

$$a_{32}x_2 + \cdots + a_{3n}x_n = x_3\lambda,$$

$$\vdots$$

$$a_{n,n-1}x_{n-1} + a_{nn}x_n = x_n\lambda.$$

From $x_n = 0$ it would follow that $x_{n-1} = x_{n-2} = \cdots = x_2 = x_1 = 0$. We try to solve these equations from the bottom upward, taking $x_n = 1$. By induction it can be proved that (with $x_n = 1$) every component x_i $(i = n-1, n-2, \ldots, 2, 1)$ is a one-sided polynomial in λ: $x_{n-1} = a_{n,n-1}^{-1}\lambda - a_{n,n-1}^{-1}a_{nn}$; $x_{n-1} = \sum_{j=0}^{i} c_{ij}\lambda^j$. Substituting these expressions into the first of the equations written out above (in place of $Ax = x\lambda$) we obtain an nth-degree one-sided polynomial equation in λ in which the coefficient of λ^n is nonzero. This proves

7.15. THEOREM. *The division ring $\mathfrak{F}$ has property $pv(n)$ if and only if every one-sided polynomial equation of degree n with coefficients in $\mathfrak{F}$ has a zero in $\mathfrak{F}$.*

The discussion that led to the above theorem did not rely on the definition of the determinant function previously given. To connect the two, we can proceed as follows.

7.16. DEFINITION. The product of the one-sided polynomials (for the various boxes) obtained above is the (strictly, a) characteristic polynomial of A.

We note that if $\mathfrak{F}$ is noncommutative, $\det(A - \lambda I)$ is not necessarily a polynomial in λ. However, we can assert

7.17. THEOREM. *Suppose every one-sided polynomial of degree n over $\mathfrak{F}$ has a zero. Then $\mathfrak{F}$ has property $pv(n)$. Moreover* $\det(A - \lambda I)$ *coincides with the characteristic polynomial of A.*

Proof. Choose P so that PAP^{-1} is triangular. Then $PAP^{-1} - \lambda I$ is also triangular. Also $P(A - \lambda I)P^{-1} = PAP^{-1} - \lambda I$. We now apply Theorem 3.8 and Definition 7.16.

7.18. THEOREM. *The determinant of a matrix is equal to the product of its proper values.*

Proof. This follows from Theorems 7.11, 3.18.

The fact that det A is defined up to multiplication by an element in the commutator group $K^{*\prime}$ of K^* is in harmony with the fact that a proper value is determined only to within conjugacy.

8. CANONICAL FORM FOR A MATRIX UNDER SIMILARITY TRANSFORMATIONS

If $\mathfrak{F}$ has property $pv(n)$, in particular if every one-sided polynomial equation has a solution in $\mathfrak{F}$, then every matrix $A \in \mathfrak{F}_n$ can be transformed into the so-called Jordan canonical form. The usual proofs of this assertion assume that $\mathfrak{F}$ is commutative, or that A is the matrix of a semilinear transformation (see [17]). In this section, we outline a different proof, based on an argument ascribed by Gel'fand to Petrovskii [11]. Next we use the properties of the determinant function to establish uniqueness.

Since the cases $n = 1, 2$ are trivial, we consider first the case $n = 3$ in detail. (The argument for general n is outlined in [11].) We suppose $A = [a_{ij}]_1{}^3$ to be in triangular form, with constant diagonal elements; see Theorem 7.12. The only difficult case is $a_{13} \neq 0$.

Case 1. Suppose first $a_{13} \neq 0$, $a_{23} = a_{12} = 0$. Then we need only permute 2, 3.

Case 2. Suppose $a_{13} \neq 0$, $a_{12} \neq 0$. We transform A by $I - a_{12}^{-1}a_{13}e_{23}$.

Case 3. Suppose $a_{13} \neq 0$, $a_{12} = 0$, $a_{23} \neq 0$. We transform by $I - a_{23}a_{13}^{-1}e_{21}$, reducing the problem to the first case.

Turning now to the case of general n, we note that by induction (on n) we may assume that $A = \begin{bmatrix} a & b \\ 0 & J \end{bmatrix}$, where J is an $n - 1 \times n - 1$ Jordan canonical form. If (first case) $a_{12} = 0$, $a_{23} = 0$, we permute 1, 2.

If (second case) $a_{12} = 0$, $a_{23} = 1$, $a_{34} = 1, \ldots, a_{r-1,r} = 1$, $a_{r,r+1} = 0$, we transform by $(0^{-1} = 0)$, $(I + a_{13}^{-1} e_{21})(I + a_{14}^{-1} e_{32}) \cdots (I + a_{1r}^{-1} e_{r-1,r})$, obtaining

$$\begin{bmatrix} a, & 0 & \cdots & 0 & & a_{1,r+1} \\ & a & 1 & & & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & a & 0 & \\ & & & & & J \end{bmatrix}$$

Then we permute the first r indices cyclically. Finally (third case) if $a_{12} \neq 0$, a long induction is needed, commencing with transformation by $\prod(I - a_{12}^{-1} a_{ij} e_{2j})$. The details are not elegant enough to warrant extensive expounding.

8.1. *Uniqueness of the Jordan canonical form.* As in the commutative case, the number of "Jordan boxes" of each dimension is an invariant. These numbers are, however, related to the elementary divisors that arise in determinant theory.

8.2. LEMMA. *If A is any $n \times n$ matrix and S is any matrix, the greatest common (polynomial) divisors of the determinants of the k-rowed minor matrices of $A - \lambda I$ and $SAS^{-1} - \lambda I$ are the same.*

The meaning of Lemma 8.2 must be explained; see below. From this lemma it follows that the Jordan canonical form is unique.

Determinants of polynomial matrices

Suppose the elements of a matrix are one-sided polynomials in a single indeterminate λ. To define the determinant of such a matrix, we invent a new object, the class of one-sided polynomials with coefficients from $K^*/K^{*\prime}$. The determinant of a polynomial matrix can now be defined as a one-sided polynomial with coefficients derived from $K^*/K^{*\prime}$, obtained by expanding the determinant of the matrix in the usual way. In fact, the coefficients may be *sums* of cosets of $K^*/K^{*\prime}$.

8.3. LEMMA. *If A is any $n \times n$ matrix and U is an elementary matrix, the greatest common divisor of the determinants of the k-rowed minor matrices of $A - \lambda I$ and $UAU^{-1} - \lambda I$ are the same polynomials.*

In fact, the minor matrices are themselves the same with only a few exceptions. In computing the gcd, constant factors are not involved, i.e., $a \sim b$ $[a, b \in K^*]$; $\lambda - a \sim b(\lambda - a)$. The proof of Lemma 8.3 depends in an obvious fashion on Lemma 4.2a.

8.4. THEOREM. *Aside from reordering of the elementary boxes, no two Jordan matrices are similar.*

This follows from Lemma 8.3 by a familiar argument [7].

There are further applications of the determinant function; the elementary symmetric functions of a transformation of a vector space can be generalized to the noncommutative case.

8.5. DEFINITION. The coefficients of the various powers of λ in the polynomial $\det(A - \lambda I)$ are the elementary symmetric functions of the matrix A.

8.6. THEOREM. *If $\mathfrak{F}$ has property $pv(n)$, the elementary symmetric functions of A are the elementary symmetric functions of the proper values of A.*

For example, the trace is a collection of cosets, and is certainly a subset of the collection

$$\{a_{11}\} + \{a_{22}\} + \cdots + \{a_{nn}\}.$$

The algebraic sum of two cosets may include elements from (and therefore be equal to the logical sum of) more than one coset.

8.7. If either A or B is invertible, AB and BA have the same characteristic polynomial.

Proof. Use Lemma 8.2 together with $BA = A^{-1}(AB)A$.

Actually much more is known. If A is $r \times m$ and B is $m \times r$, the nonzero proper values of AB and BA coincide. This is established by the following little-known computation. Assume $m > r$.

8.8. THEOREM. *The proper values of BA are the same as those of AB, together with $m - r$ zeros.*

Proof. We begin with the equations

$$\begin{bmatrix} \lambda I & 0 \\ -A & \lambda I \end{bmatrix}\begin{bmatrix} \lambda I & B \\ A & \lambda I \end{bmatrix}\begin{bmatrix} \lambda I & -B \\ 0 & \lambda I \end{bmatrix} = \begin{bmatrix} \lambda^3 I_m & 0 \\ 0 & \lambda^3 I_r - \lambda AB \end{bmatrix},$$

$$\begin{bmatrix} \lambda I & -B \\ 0 & \lambda I \end{bmatrix}\begin{bmatrix} \lambda I & B \\ A & \lambda I \end{bmatrix}\begin{bmatrix} \lambda I & 0 \\ -A & \lambda I \end{bmatrix} = \begin{bmatrix} \lambda^3 I_m - \lambda BA & 0 \\ 0 & \lambda^3 I_r \end{bmatrix}.$$

By property 1.5, the right members of these relations have the same determinant. Using Theorem 3.16, we find $\lambda^{3m+r}\det(\lambda^2 I_r - AB) = \lambda^{3r+m}\det(\lambda^2 I_m - BA)$; thus $\mu^{m-r}\det(\mu I_r - AB) = \det(\mu I_m - BA)$, where $\mu = \lambda^2$. This is a relation involving the indeterminate μ. Thus BA has $m - r$ more zero proper values than does AB. A similar, slightly more complicated computation [1, p. 371] can be used to obtain the known relations [10] among the elementary divisors of AB, BA.

9. KRONECKER PRODUCTS

9.1. DEFINITION OF $I_m \times A$. Let A be an $n \times n$ matrix. Let I_m be the $m \times m$ identity matrix. The object $I_m \times A$ is an $mn \times mn$ partitioned matrix, in which the $m \times m$ boxes are scalar matrices. The (i, j) box is $a_{ij} I_m$, i.e., the $m \times m$ scalar matrix with diagonal element a_{ij} (the i, j element of A).

9.2. DEFINITION OF $B \times I_n$. Let B be an $m \times m$ matrix. The object $B \times I_n$ is an $mn \times mn$ partitioned matrix in which the $n \times n$ boxes are all zero except the diagonal ones, which are all B: $B \times I_n = B \oplus B \oplus \cdots \oplus B$ (n summands).

9.3. LEMMA. *$I_m \times A$ can be transformed into $A \times I_m$ by a permutation.*

9.4. LEMMA. $\det(I_m \times A) = (\det A)^m$. $\det(B \times I_n) = (\det B)^n$.

Proof. Theorem 3.16.

9.5. Let A be $n \times n$; B, $m \times m$. The object $A \times B$ (Kronecker product) is defined as $(I_m \times A) \cdot (B \times I_n)$, i.e., the matrix product of these two $mn \times mn$ matrices.

9.6. THEOREM (Givens). $\det(A \times B) = (\det A)^m(\det B)^n$.

Without using Lemma 9.3, the proof of which is tedious, we can arrive at the same result by using the hybrid theorem 6.1 to establish that

$$\det(I_m \times A) = \det \begin{bmatrix} \det A & & & \\ & \det A & & \\ & & \det A & \\ & & & \ddots \end{bmatrix} = (\det A)^m.$$

This proof seems quite easily comprehended and direct.

10. ROOT-LOCATION THEOREMS

For matrices of quaternions, it makes sense to speak not only of proper values, but also of their absolute values. A good deal of the wide literature on root location carries over to this noncommutative domain. An overview of some of these theorems is given in [6]. The following single example is interesting because it involves the determinant function.

10.1. THEOREM. *Let $A = [a_{ij}]$ be an $n \times n$ matrix; let each of the indices i ($i = 1, 2, \ldots, n$) be contained in a subset $J(i)$ of these indices. Then every proper value of A is contained in one of the loci (satisfies at least one of the relations), $i = 1, \ldots, n$,*

$$\left|\det B\begin{pmatrix} J(i) \\ J(i) \end{pmatrix}\right| \leqslant \sum_{\nu, \nu \notin J(i)} \left|\det B\begin{pmatrix} J(i) \\ J(i)\setminus i,\ \nu \end{pmatrix}\right|$$

where $B = A - \lambda I$; $B\begin{pmatrix} J(i) \\ J(i) \end{pmatrix}$ is the matrix on rows $\{J(i)\}$ and columns $\{J(i)\}$; $B\begin{pmatrix} J(i) \\ J(i)\setminus i,\ \nu \end{pmatrix}$ is the matrix on rows $\{J(i)\}$ and columns $J(i)$ with i omitted and ν appended. The number of loci or relations is precisely n.

The proof uses exterior algebra, in particular Theorem 5.2. See [6] for details, and note that $|q| = 1$ if q is a commutator of quaternions.

11. PERMANENTS

If $\mathfrak{F}$ is commutative, the permanent of $A \in \mathfrak{F}_n$ is usually defined as the multilinear form $\sum a_{1\sigma(1)}a_{2\sigma(2)} \cdots a_{n\sigma(n)}$, the summation being extended over all $n!$ permutations σ of the indices. In our view this definition should be extended to a noncommutative domain by starting with the determinant function, as follows.

11.1. DEFINITION. Let $A = [a_{ij}]$ be an $n \times n$ matrix with elements in the division ring $\mathfrak{F}$. If $\det A$ can be written in the form

$$\det A = \sum (-1)^{\sigma} a_{1\sigma(1)} \cdots a_{n\sigma(n)} \cdot w_{\sigma}$$

where the summation is extended over all possible $n!$ permutations of the indices, and where w_{σ} is a commutator of the multiplicative group of $\mathfrak{F}$, then the sum $\sum a_{1\sigma(1)} \cdots a_{n\sigma(n)} w'_{\sigma}$ is a coset in per A. Per A consists of all cosets that can be represented in this way.

We do not pursue this definition very far. Although at first glance the function seems to have few properties, L. Beasley has obtained some results concerning it (unpublished). We also point out

11.2. THEOREM. *The permanent of a matrix may be evaluated by expanding by minors in the way usual for permanents.*

12. FURTHER QUESTIONS

The potpourri of results in this paper indicates the possibility that other useful extensions of commutative geometry to the noncommutative case may be accessible through the use of the Dieudonné determinant. The field of real quaternions can be *valued*: there is an automorphism $(*)$ such that $\alpha\alpha^* = |\alpha|^2$. Intricate theorems concerning positive definite hermitian forms (see [8]) can therefore probably be extended to quaternion matrices. (*Added in proof*: This has been done by De Pillis and the author.)

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